

Continuity bounds for information characteristics of quantum channels depending on input dimension and on input energy

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Abstract

We obtain continuity bounds for basic information characteristics of quantum channels depending on their input dimension (when it is finite) and on the maximal level of input energy (when the input dimension is infinite). We pay a special attention to the case when the input system is the multi-mode quantum oscillator.

First we prove continuity bounds for the output conditional mutual information for a single channel and for n copies of a channel.

Then we obtain estimates for variation of the output Holevo quantity with respect to simultaneous variations of a channel and of an input ensemble.

As a result tight and close-to-tight continuity bounds for basic capacities of quantum channels depending on the input dimension are obtained. They complement the Leung-Smith continuity bounds depending on the output dimension.

Finally, continuity bounds for basic capacities of infinite-dimensional channels with the input energy constraints are proved. They show uniform continuity of these capacities on the set of all channels with respect to the norm of complete boundedness.

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1 Introduction

Leung and Smith obtained in [13] continuity bounds (estimates for variations) for basic capacities of quantum channels depending of their output dimension. The appearance of the output dimension in these and some other continuity bounds for information characteristics of a quantum channel is natural, since such characteristics are typically expressed via entropic quantities of output states of a channel (so, application of Fannes' type continuity bounds gives the factor proportional to the logarithm of the output dimension [1, 2, 5, 29]).

At the same time, it is the input dimension of a channel that determines the range of its information capacities, while the formal output dimension may be substantially greater than the real dimension of a channel output. So, it is reasonable to conjecture that the input dimension also determines the rate of variations of capacities regardless of the output dimension.

Speaking about capacities of channels between infinite-dimensional quantum systems we have to impose energy constraints on states used for coding information [7, 8, 27]. In this case the range of information capacities is determined by the maximal level of (average) energy of states-codes, which plays, roughly speaking, the role of input dimension. So, we may expect that the maximal level of input energy also determines the rate of variations of capacities and of other entropic characteristics of infinite-dimensional quantum channels.

In this paper we confirm both these conjectures by deriving continuity bounds for several information characteristics of quantum channels depending on their input dimension (when it is finite) and on the maximal level of input energy (when the input dimension is infinite).

We will consider the case of finite input dimension and the case of finite input energy simultaneously excepting the last Sections 6 and 7 devoted, respectively, to the first and to the second cases.

We begin with proving special continuity bounds for the quantum conditional mutual information (Lemma 3 in Section 3), which is the main technical tool in this paper.

In Section 4 we obtain continuity bounds for the output conditional mutual information $I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$ with respect to simultaneous variations of a channel $\Phi : A \rightarrow B$ and of an input state ρ_{ACD} . We also derive continuity bound for the function $\Phi \mapsto I(B^n:D|C)_{\Phi^{\otimes n} \otimes \text{Id}_{CD}(\rho)}$ for any natural n by using the Leung-Smith telescopic trick.

In Section 5 we analyse continuity properties of the output Holevo quantity $\chi(\Phi(\mu))$ – the Holevo quantity of the ensemble $\Phi(\mu)$ obtained by action of a channel Φ on a given (discrete or continuous) ensemble μ of input states. We obtain estimates for variation of $\chi(\Phi(\mu))$ with respect to simultaneous variations of a channel Φ and of an input ensemble μ .

In Section 6 the above results are applied to obtain tight and close-to-tight continuity bounds for basic capacities of quantum channels depending on their input dimension. They complement the above-mentioned Leung-Smith continuity bounds (depending on the output dimension).

In Section 7 continuity bounds for basic capacities of infinite-dimensional channels with the input energy constraints are proved. They show uniform continuity of these capacities on the set of *all* channels with respect to the norm of complete boundedness provided that the Hamiltonian of input system satisfies the particular condition. This condition (relation (19) in Section 2.2.1) is nonrestrictive, it holds, for example, if input system is the multi-mode quantum oscillator. This case playing important role in quantum information theory (cf.[7, 24]) is considered separately after formulations of general results.

2 Preliminaries

2.1 Basic notations and auxiliary lemmas

Let \mathcal{H} be a finite-dimensional or separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators with the operator norm $\|\cdot\|$ and $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators in \mathcal{H} with the trace norm $\|\cdot\|_1$. Let $\mathfrak{S}(\mathcal{H})$ be the set of quantum states (positive operators in $\mathfrak{T}(\mathcal{H})$ with unit trace) [7, 26].

Denote by $I_{\mathcal{H}}$ the unit operator in a Hilbert space \mathcal{H} and by $\text{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

If quantum systems A and B are described by Hilbert spaces \mathcal{H}_A and \mathcal{H}_B then the bipartite system AB is described by the tensor product of these spaces, i.e. $\mathcal{H}_{AB} \doteq \mathcal{H}_A \otimes \mathcal{H}_B$. A state in $\mathfrak{S}(\mathcal{H}_{AB})$ is denoted ρ_{AB} , its marginal states $\text{Tr}_B \rho_{AB}$ and $\text{Tr}_A \rho_{AB}$ are denoted respectively ρ_A and ρ_B .¹

¹Here and in what follows Tr_X means $\text{Tr}_{\mathcal{H}_X}$.

A *quantum channel* Φ from a system A to a system B is a completely positive trace preserving linear map from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_B)$ [7, 26].

For any quantum channel $\Phi : A \rightarrow B$ the Stinespring theorem implies existence of a Hilbert space \mathcal{H}_E and of an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_E V \rho V^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \quad (1)$$

The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni \rho \mapsto \widehat{\Phi}(\rho) = \text{Tr}_B V \rho V^* \in \mathfrak{T}(\mathcal{H}_E) \quad (2)$$

is called *complementary* to the channel Φ [7, Ch.6].

The set of quantum channels is typically equipped with the metric induced by the *diamond norm*

$$\|\Phi\|_\diamond \doteq \sup_{\rho \in \mathfrak{T}(\mathcal{H}_A), \|\rho\|_1=1} \|\Phi \otimes \text{Id}_R(\rho)\|_1 \quad (3)$$

on the set of completely bounded linear maps from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_B)$. This norm coincides with the norm of complete boundedness of the dual map $\Phi^* : \mathfrak{B}(\mathcal{H}_B) \rightarrow \mathfrak{B}(\mathcal{H}_A)$ to the map Φ [7, 13, 26].

For our purposes it is more convenient to use the equivalent metric on the set of channels called *Bures distance* defined for given channels $\Phi : A \rightarrow B$ and $\Psi : A \rightarrow B$ as follows (cf.[12])

$$\beta(\Phi, \Psi) = \inf \|V_\Phi - V_\Psi\|, \quad (4)$$

where the infimum is over all common Stinespring representations:

$$\Phi(\rho) = \text{Tr}_E V_\Phi \rho V_\Phi^* \quad \text{and} \quad \Psi(\rho) = \text{Tr}_E V_\Psi \rho V_\Psi^*. \quad (5)$$

Theorem 1 in [12] states that the infimum in (4) is attainable and that

$$\frac{1}{2} \|\Phi - \Psi\|_\diamond \leq \beta(\Phi, \Psi) \leq \sqrt{\|\Phi - \Psi\|_\diamond}, \quad (6)$$

which shows the equivalence of the Bures distance and the diamond norm distance on the set of all channels between given quantum systems.

The *von Neumann entropy* $H(\rho) = \text{Tr} \eta(\rho)$ of a state $\rho \in \mathfrak{S}(\mathcal{H})$, where $\eta(x) = -x \log x$, is a concave nonnegative lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$, it is continuous if and only if $\dim \mathcal{H} < +\infty$ [7, 17, 26].

The *quantum relative entropy* for two states ρ and σ in $\mathfrak{S}(\mathcal{H})$ is defined as follows

$$H(\rho\|\sigma) = \sum_i \langle i | \rho \log \rho - \rho \log \sigma | i \rangle,$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of the state ρ and it is assumed that $H(\rho\|\sigma) = +\infty$ if the support of ρ is not contained in the support of σ [7, 17].²

The *quantum mutual information* of a state ρ_{AB} of a bipartite quantum system is defined as follows

$$I(A:B)_\rho = H(\rho_{AB}\|\rho_A \otimes \rho_B) = H(\rho_A) + H(\rho_B) - H(\rho_{AB}), \quad (7)$$

where the second expression is valid if $H(\rho_{AB})$ is finite [16, 26].

Basic properties of the relative entropy show that $\rho \mapsto I(A:B)_\rho$ is a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{AB})$ taking values in $[0, +\infty]$. It is well known that

$$I(A:B)_\rho \leq 2 \min \{H(\rho_A), H(\rho_B)\} \quad (8)$$

for any state ρ_{AB} and that

$$I(A:B)_\rho \leq \min \{H(\rho_A), H(\rho_B)\} \quad (9)$$

for any separable state ρ_{AB} [14, 26].

The *quantum conditional mutual information* of a state ρ_{ABC} of a tripartite finite-dimensional system is defined as follows

$$I(A:B|C)_\rho \doteq H(\rho_{AC}) + H(\rho_{BC}) - H(\rho_{ABC}) - H(\rho_C). \quad (10)$$

This quantity plays important role in quantum information theory [4, 26], its nonnegativity is a basic result well known as *strong subadditivity of von Neumann entropy* [15]. If system C is trivial then (10) coincides with (7).

In infinite dimensions formula (10) may contain the uncertainty " $\infty - \infty$ ". Nevertheless the conditional mutual information can be defined for any state ρ_{ABC} by one of the equivalent expressions

$$I(A:B|C)_\rho = \sup_{P_A} [I(A:BC)_{Q_A \rho Q_A} - I(A:C)_{Q_A \rho Q_A}], \quad Q_A = P_A \otimes I_{BC}, \quad (11)$$

²The support of a positive operator is the orthogonal complement to its kernel.

$$I(A:B|C)_\rho = \sup_{P_B} [I(B:AC)_{Q_B \rho Q_B} - I(B:C)_{Q_B \rho Q_B}], \quad Q_B = P_B \otimes I_{AC}, \quad (12)$$

where the suprema are over all finite rank projectors $P_A \in \mathfrak{B}(\mathcal{H}_A)$ and $P_B \in \mathfrak{B}(\mathcal{H}_B)$ correspondingly and it is assumed that $I(X:Y)_{Q_X \rho Q_X} = \lambda I(X:Y)_{\lambda^{-1} Q_X \rho Q_X}$, where $\lambda = \text{Tr} Q_X \rho_{ABC}$ [21].

Expressions (11) and (12) define the same lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{ABC})$ possessing all basic properties of the quantum conditional mutual information valid in finite dimensions [21, Th.2]. In particular, the following relation (chain rule)

$$I(X:YZ|C)_\rho = I(X:Y|C)_\rho + I(X:Z|YC)_\rho \quad (13)$$

holds for any state ρ in $\mathfrak{S}(\mathcal{H}_{XYZC})$ (with possible values $+\infty$ in both sides). To prove (13) is suffices to note that it holds if the systems X, Y, Z and C are finite-dimensional and to apply Corollary 9 in [21].

We will also use the upper bound

$$I(A:B|C)_\rho \leq 2 \min \{H(\rho_A), H(\rho_B), H(\rho_{AC}), H(\rho_{BC})\} \quad (14)$$

valid for any state ρ_{ABC} . It directly follows from upper bound (8) and the expression $I(X:Y|C)_\rho = I(X:YC)_\rho - I(X:C)_\rho$, $X, Y = A, B$, which is a partial case of (13).

The quantum conditional mutual information is not concave or convex but the following relation

$$|pI(A:B|C)_\rho + (1-p)I(A:B|C)_\sigma - I(A:B|C)_{p\rho+(1-p)\sigma}| \leq h_2(p) \quad (15)$$

holds for $p \in (0, 1)$ and any states ρ_{ABC}, σ_{ABC} with finite $I(A:B|C)_\rho, I(A:B|C)_\sigma$, where $h_2(p) = \eta(p) + \eta(1-p)$ is the binary entropy [22].

We will repeatedly use the following simple lemma.

Lemma 1. *If U and V are isometries from \mathcal{H} into \mathcal{H}' then*

$$\|U\rho U^* - V\rho V^*\|_1 \leq 2\|U - V\|$$

for any state ρ in $\mathfrak{S}(\mathcal{H})$.

2.2 Set of quantum states with bounded energy

In this subsection we briefly describe some properties of sets of quantum states with bounded mean energy playing the role of "space of states" in analysis of infinite-dimensional quantum systems and channels.

2.2.1 General case

Let H_A be a positive operator in a Hilbert space \mathcal{H}_A treated as a Hamiltonian of quantum system A . Let $E \geq E_0 \doteq \inf_{\|\varphi\|=1} \langle \varphi | H_A | \varphi \rangle$. Then

$$\mathfrak{C}_{H_A, E} = \{\rho \in \mathfrak{S}(\mathcal{H}_A) \mid \text{Tr} H_A \rho \leq E\}$$

is a closed convex subset of $\mathfrak{S}(\mathcal{H}_A)$ consisting of states with mean energy not exceeding E .

It is well known that the von Neumann entropy is continuous on the set $\mathfrak{C}_{H_A, E}$ for any $E \geq E_0$ if (and only if) the Hamiltonian H_A satisfies the condition

$$\text{Tr} e^{-\lambda H_A} < +\infty \quad \text{for all } \lambda > 0 \quad (16)$$

and that it achieves the maximal value on this set at the *Gibbs state* $\gamma_A(E) \doteq e^{-\lambda(E)H_A} / \text{Tr} e^{-\lambda(E)H_A}$, where the parameter $\lambda(E)$ is determined by the equality $\text{Tr} H_A e^{-\lambda(E)H_A} = E \text{Tr} e^{-\lambda(E)H_A}$ [25].

Condition (16) implies that the operator H_A has a discrete spectrum of finite multiplicity, i.e. it can be represented as follows $H_A = \sum_{k=0}^{+\infty} E_k |\tau_k\rangle\langle\tau_k|$, where $\{E_k\}$ is the nondecreasing sequence of eigenvalues of H_A and $\{|\tau_k\rangle\}$ – the corresponding basis of eigenvectors.

In what follows we will use the function

$$F_{H_A}(E) \doteq \sup_{\rho \in \mathfrak{C}_{H_A, E}} H(\rho) = H(\gamma_A(E)).$$

Properties of this function are described in Proposition 1 in [20], where it shown, in particular, that

$$F_{H_A}(E) = \lambda(E)E + \log \text{Tr} e^{-\lambda(E)H_A} = o(E) \quad \text{as } E \rightarrow +\infty, \quad (17)$$

and that

$$F'_{H_A}(E) = \lambda(E) > 0, \quad F''_{H_A}(E) = \lambda'(E) < 0$$

for all $E > E_0$ provided that condition (16) holds. It means that F_{H_A} is a strictly increasing concave function on $[E_0, +\infty)$ such that $F_{H_A}(E_0) = \log d_0$, where d_0 is the multiplicity of the eigenvalue E_0 .

Let $F_{H_A}^{-1}$ be the inverse function to the function F_{H_A} . The above-stated properties of F_{H_A} shows that $F_{H_A}^{-1}$ is an increasing function on $[\log d_0, +\infty)$ taking values in $[E_0, +\infty)$.

Lemma 2. Let $E \geq E_0$, $d \geq d_0$ and $\gamma(d) \doteq F_{H_A}^{-1}(\log d)$.

A) For any $\rho \in \mathfrak{C}_{H_A, E}$ there is a state $\sigma \in \mathfrak{C}_{H_A, E}$ such that

$$\text{rank } \sigma \leq d \quad \text{and} \quad \frac{1}{2} \|\rho - \sigma\|_1 \leq E/\gamma(d).$$

B) Let $\mathfrak{C}_{H_A, E}^{\text{ext}} \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_{AB}) \mid \rho_A \in \mathfrak{C}_{H_A, E}\}$, where B is a given system. Then for any $\rho \in \mathfrak{C}_{H_A, E}^{\text{ext}}$ there is a state $\sigma \in \mathfrak{C}_{H_A, E}^{\text{ext}}$ such that

$$\text{rank } \sigma_A \leq d \quad \text{and} \quad \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{2E/\gamma(d)}.$$

If ρ is a pure state then the corresponding state σ can be taken pure.

Remark 1. Since $\gamma(d) \rightarrow +\infty$ as $d \rightarrow +\infty$, Lemma 2 shows that the set $\mathfrak{C}_{H_A, E}$ (correspondingly, the set $\mathfrak{C}_{H_A, E}^{\text{ext}}$) can be uniformly approximated by its subsets consisting of states with bounded rank (correspondingly, of states with bounded rank of partial state). This fact can be proved by using the compactness of the set $\mathfrak{C}_{H_A, E}$ (see Lemma 1 in [23]), but Lemma 2 gives the explicit estimates for the accuracy of such approximation.

Proof. It suffices to consider the case $E \leq \gamma(d)$.

A) Note first that

$$C \doteq \max_{1 \leq k \leq d} \{\langle k | H_A | k \rangle\} \geq \gamma(d) \tag{18}$$

for any orthonormal set $\{|k\rangle\}_{k=1}^d \subset \mathcal{H}_A$. Indeed, if $P_d = \sum_{k=1}^d |k\rangle\langle k|$ then $\text{Tr} P_d H_A \leq Cd$. So, the state $d^{-1}P_d$ belongs to the set $\mathfrak{C}_{H_A, C}$ and hence its entropy $\log d$ does not exceed $F_{H_A}(C)$.

Let $\rho = \sum_{k=1}^{+\infty} p_k |k\rangle\langle k|$ be a state in $\mathfrak{C}_{H_A, E}$. We may assume that the basis $\{|k\rangle\}$ is reordered in such a way that the sequence $\{\langle k | H_A | k \rangle\}_k$ is non-decreasing.

Let $\sigma = (1 - \delta_d)^{-1} \sum_{k=1}^d p_k |k\rangle\langle k|$, where $\delta_d = \sum_{k>d} p_k$. Since $\rho \in \mathfrak{C}_{H_A, E}$, we have $\sum_{k>d} p_k \langle k | H_A | k \rangle \leq E$ and hence

$$\delta_d \leq E/\langle d | H_A | d \rangle \leq E/\gamma(d) \leq 1,$$

where the second inequality follows from (18). It is easy to see that the inequality $E/\langle d | H_A | d \rangle \leq 1$ implies $\text{Tr} H_A \sigma \leq E$ and that $\|\rho - \sigma\|_1 \leq 2\delta_d$.

B) By using definition of the fidelity F and its monotonicity under partial trace it is easy to show that for any states ρ_{AB} and σ_A there is an extension σ_{AB} of σ_A such that $F(\rho_{AB}, \sigma_{AB}) = F(\rho_A, \sigma_A)$. If the state ρ_{AB} is pure then

the extension σ_{AB} can be taken pure as well. These observations and the well known relations

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}$$

between the fidelity and the trace norm (cf.[7, 26]) make it possible to derive this assertion of the lemma from the assertion A. \square

In this paper we will essentially use the modification of the Alicki-Fannes-Winter method³ adapted for the set of states with bounded energy [23]. This modification makes it possible to prove uniform continuity of any locally almost affine function⁴ f on the set

$$\mathfrak{C}_{H_A, E}^{\text{ext}} \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_{AB}) \mid \rho_A \in \mathfrak{C}_{H_A, E}\} \quad (B \text{ is any given system})$$

such that $|f(\rho_{AB})| \leq CH(\rho_A)$ for some $C \in \mathbb{R}_+$ provided that

$$\lim_{\lambda \rightarrow +0} [\text{Tr} e^{-\lambda H_A}]^\lambda = 1. \quad (19)$$

Condition (19) implies $F_{H_A}(E) = o(\sqrt{E})$ as $E \rightarrow +\infty$ (cf. (17)). It is stronger than condition (16) but the difference between these conditions is not too large. In terms of the sequence $\{E_k\}$ of eigenvalues of H_A condition (16) means that $\lim_{k \rightarrow \infty} E_k / \log k = +\infty$, while condition (19) is valid if $\liminf_{k \rightarrow \infty} E_k / \log^q k > 0$ for some $q > 2$ [23, Pr.1].

It is essential that condition (19) holds for the Hamiltonian of the system of quantum oscillators playing central role in continuous variable quantum information theory [24].

2.2.2 The ℓ -mode quantum oscillator

Consider now the case when system A is the ℓ -mode quantum oscillator. In this case

$$H_A = \sum_{i=1}^{\ell} \hbar \omega_i (a_i^\dagger a_i + \frac{1}{2} I_A),$$

³This method is widely used in finite-dimensions for proving uniform continuity of functions on the set of quantum states [1, 29].

⁴This means that $|f(p\rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma)| \leq h(p) = o(1)$ for $p \rightarrow +0$.

where a_i and a_i^+ are the annihilation and creation operators and ω_i is the frequency of the i -th oscillator [7, Ch.12]. It follows that

$$F_{H_A}(E) = \max_{\{E_i\}} \sum_{i=1}^{\ell} g(E_i/\hbar\omega_i - 1/2), \quad E \geq E_0 \doteq \frac{1}{2} \sum_{i=1}^{\ell} \hbar\omega_i,$$

where $g(x) = (x+1)\log(x+1) - x\log x$ and the maximum is over all ℓ -tuples E_1, \dots, E_ℓ such that $\sum_{i=1}^{\ell} E_i = E$ and $E_i \geq \frac{1}{2}\hbar\omega_i$. The exact value of $F_{H_A}(E)$ can be calculated by applying the Lagrange multiplier method which leads to a transcendental equation. But one can obtain tight upper bound for $F_{H_A}(E)$ by using the inequality $g(x) \leq \log(x+1) + 1$ (cf. [29]). It implies

$$F_{H_A}(E) \leq \max_{\sum_{i=1}^{\ell} E_i = E} \sum_{i=1}^{\ell} \log(E_i/\hbar\omega_i + 1/2) + \ell.$$

By calculating this maximum via the Lagrange multiplier method we obtain

$$F_{H_A}(E) \leq \widehat{F}_{\ell,\omega}(E) \doteq \ell \log \frac{E + E_0}{\ell E_*} + \ell, \quad E_* = \left[\prod_{i=1}^{\ell} \hbar\omega_i \right]^{1/\ell}. \quad (20)$$

Since $g(x) = \log(x+1) + 1 + o(1)$ for large x , upper bound (20) is tight for large E .

By using the function $\widehat{F}_{\ell,\omega}$ one can define the sequence of positive numbers

$$\hat{\gamma}(d) \doteq \widehat{F}_{\ell,\omega}^{-1}(\log d) = (\ell/e)E_* \sqrt[\ell]{d} - E_0, \quad d > e^{\widehat{F}_{\ell,\omega}(0)}, \quad (21)$$

which can be used instead of the sequence $\gamma(d)$ in Lemma 2. It follows from (20) that $\hat{\gamma}(d) \leq \gamma(d)$ for all d .

2.3 Some facts about ensembles of quantum states

2.3.1 Discrete ensembles

A finite or countable collection $\{\rho_i\}$ of states with a probability distribution $\{p_i\}$ is conventionally called *discrete ensemble* and denoted $\{p_i, \rho_i\}$. The state $\bar{\rho} \doteq \sum_i p_i \rho_i$ is called *average state* of this ensemble.

The *Holevo quantity* of an ensemble $\{p_i, \rho_i\}_{i=1}^m$ of $m \leq +\infty$ quantum states is defined as

$$\chi(\{p_i, \rho_i\}_{i=1}^m) \doteq \sum_{i=1}^m p_i H(\rho_i \| \bar{\rho}) = H(\bar{\rho}) - \sum_{i=1}^m p_i H(\rho_i),$$

where the second formula is valid if $H(\bar{\rho}) < +\infty$. This quantity gives the upper bound for classical information obtained by recognizing states of the ensemble by quantum measurements [6]. It plays important role in analysis of information properties of quantum systems and channels [7, 26].

Let $\mathcal{H}_A = \mathcal{H}$ and $\{|i\rangle\}_{i=1}^m$ be an orthonormal basis in a m -dimensional Hilbert space \mathcal{H}_B . Then it is easy to show that

$$\chi(\{p_i, \rho_i\}_{i=1}^m) = I(A:B)_{\hat{\rho}}, \text{ where } \hat{\rho}_{AB} = \sum_{i=1}^m p_i \rho_i \otimes |i\rangle\langle i|. \quad (22)$$

The state $\hat{\rho}_{AB}$ is called *qc-state* determined by the ensemble $\{p_i, \rho_i\}_{i=1}^m$ [26].

In analysis of continuity of the Holevo quantity we will use three different measures of divergence between ensembles $\mu = \{p_i, \rho_i\}$ and $\nu = \{q_i, \sigma_i\}$.

The quantity

$$D_0(\mu, \nu) \doteq \frac{1}{2} \sum_i \|p_i \rho_i - q_i \sigma_i\|_1$$

is a true metric on the set of all ensembles of quantum states considered as *ordered* collections of states with the corresponding probability distributions. It coincides (up to the factor 1/2) with the trace norm of the difference between the corresponding *qc-states* $\sum_i p_i \rho_i \otimes |i\rangle\langle i|$ and $\sum_i q_i \sigma_i \otimes |i\rangle\langle i|$.

The main advantage of D_0 is a direct computability, but from the quantum information point of view we have to consider an ensemble of quantum states $\{p_i, \rho_i\}$ as a discrete probability measure $\sum_i p_i \delta(\rho_i)$ on the set $\mathfrak{S}(\mathcal{H})$ (where $\delta(\rho)$ is the Dirac measure concentrating at a state ρ) rather than ordered (or disordered) collection of states. If we want to identify ensembles corresponding to the same probability measure then it is natural to use the factorization of D_0 , i.e. the quantity

$$D_*(\mu, \nu) \doteq \inf_{\mu' \in \mathcal{E}(\mu), \nu' \in \mathcal{E}(\nu)} D_0(\mu', \nu') \quad (23)$$

as a measure of divergence between ensembles $\mu = \{p_i, \rho_i\}$ and $\nu = \{q_i, \sigma_i\}$, where $\mathcal{E}(\mu)$ and $\mathcal{E}(\nu)$ are the sets of all countable ensembles corresponding to the measures $\sum_i p_i \delta(\rho_i)$ and $\sum_i q_i \delta(\sigma_i)$ respectively.

It is shown in [22] that the factor-metric D_* coincides with the EHS-distance D_{ehs} between ensembles of quantum states proposed by Oreshkov and

Calsamiglia in [18] and that D_* generates the weak convergence topology on the set of all ensembles (considered as probability measures).⁵

The metric $D_* = D_{\text{ehs}}$ is more adequate for continuity analysis of the Holevo quantity, but difficult to compute in general.⁶ It is clear that

$$D_*(\mu, \nu) \leq D_0(\mu, \nu) \quad (24)$$

for any ensembles μ and ν . But in some cases the metrics D_0 and D_* is close to each other or even coincide. This holds, for example, if we consider small perturbations of states or probabilities of a given ensemble.

The third useful metric is the Kantorovich distance

$$D_K(\mu, \nu) = \frac{1}{2} \inf_{\{P_{ij}\}} \sum P_{ij} \|\rho_i - \sigma_j\|_1 \quad (25)$$

between ensembles $\mu = \{p_i, \rho_i\}$ and $\nu = \{q_i, \sigma_i\}$, where the infimum is over all joint probability distributions $\{P_{ij}\}$ with the marginals $\{p_i\}$ and $\{q_i\}$, i.e. such that $\sum_j P_{ij} = p_i$ for all i and $\sum_i P_{ij} = q_j$ for all j . By noting that $D_* = D_{\text{ehs}}$, it is easy to show (see [18]) that

$$D_*(\mu, \nu) \leq D_K(\mu, \nu) \quad (26)$$

for any discrete ensembles μ and ν .

The main advantage of the Kantorovich distance is the existence of its natural extension to the set of all generalized (continuous) ensembles which generates the weak convergence topology on this set (see the next subsection).

If μ and ν are discrete ensembles of states in $\mathfrak{S}(\mathcal{H})$, where $d \doteq \dim \mathcal{H} < +\infty$, then Proposition 5 in [22] implies that

$$|\chi(\mu) - \chi(\nu)| \leq \varepsilon \log d + 2g(\varepsilon), \quad (27)$$

where $\varepsilon = D_*(\mu, \nu)$ and $g(\varepsilon) = (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1+\varepsilon}\right)$. Since $g(\varepsilon)$ is an increasing function, it follows from (24) and (26) that inequality (27) remains valid for $\varepsilon = D_0(\mu, \nu)$ and for $\varepsilon = D_K(\mu, \nu)$.⁷

⁵This means that a sequence $\{\{p_i^n, \rho_i^n\}\}_n$ converges to an ensemble $\{p_i^0, \rho_i^0\}$ with respect to the metric D_* if and only if $\lim_{n \rightarrow \infty} \sum_i p_i^n f(\rho_i^n) = \sum_i p_i^0 f(\rho_i^0)$ for any continuous bounded function f on $\mathfrak{S}(\mathcal{H})$.

⁶For finite ensembles it can be calculated by a linear programming procedure [18].

⁷Continuity bound (27) with $\varepsilon = D_K(\mu, \nu)$ is a refinement of the continuity bound for the Holevo quantity obtained by Oreshkov and Calsamiglia in [18].

2.3.2 Generalized (continuous) ensembles

In analysis of infinite-dimensional quantum systems and channels the notion of *generalized (continuous) ensemble* defined as a Borel probability measure on the set of quantum states naturally appears [7, 9]. We denote by $\mathcal{P}(\mathcal{H})$ the set of all Borel probability measures on $\mathfrak{S}(\mathcal{H})$ equipped with the topology of weak convergence [3, 19].⁸ The set $\mathcal{P}(\mathcal{H})$ is a complete separable metric space containing the dense subset $\mathcal{P}_0(\mathcal{H})$ of discrete measures (corresponding to discrete ensembles) [19]. The average state of a generalized ensemble $\mu \in \mathcal{P}(\mathcal{H})$ is the barycenter of the measure μ defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \rho \mu(d\rho).$$

For an ensemble $\mu \in \mathcal{P}(\mathcal{H}_A)$ its image $\Phi(\mu)$ under a quantum channel $\Phi : A \rightarrow B$ is defined as the ensemble in $\mathcal{P}(\mathcal{H}_B)$ corresponding to the measure $\mu \circ \Phi^{-1}$ on $\mathfrak{S}(\mathcal{H}_B)$, i.e. $\Phi(\mu)[\mathfrak{S}_B] = \mu[\Phi^{-1}(\mathfrak{S}_B)]$ for any Borel subset $\mathfrak{S}_B \subseteq \mathfrak{S}(\mathcal{H}_B)$, where $\Phi^{-1}(\mathfrak{S}_B)$ is the pre-image of \mathfrak{S}_B under the map Φ . If $\mu = \{p_i, \rho_i\}$ then this definition implies $\Phi(\mu) = \{p_i, \Phi(\rho_i)\}$.

The Holevo quantity of a generalized ensemble $\mu \in \mathcal{P}(\mathcal{H})$ is defined as

$$\chi(\mu) = \int H(\rho \| \bar{\rho}(\mu)) \mu(d\rho) = H(\bar{\rho}(\mu)) - \int H(\rho) \mu(d\rho),$$

where the second formula is valid under the condition $H(\bar{\rho}(\mu)) < +\infty$ [9].

The Kantorovich distance (25) between discrete ensembles is extended to generalized ensembles μ and ν as follows

$$D_K(\mu, \nu) = \frac{1}{2} \inf_{\Lambda \in \Pi(\mu, \nu)} \int_{\mathfrak{S}(\mathcal{H}) \times \mathfrak{S}(\mathcal{H})} \|\rho - \sigma\|_1 \Lambda(d\rho, d\sigma), \quad (28)$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on $\mathfrak{S}(\mathcal{H}) \times \mathfrak{S}(\mathcal{H})$ with the marginals μ and ν . Since $\frac{1}{2} \|\rho - \sigma\|_1 \leq 1$ for all ρ and σ , the Kantorovich distance (28) generates the weak convergence topology on $\mathcal{P}(\mathcal{H})$ [3].

⁸The weak convergence of a sequence $\{\mu_n\}$ to a measure μ_0 means that $\lim_{n \rightarrow \infty} \int f(\rho) \mu_n(d\rho) = \int f(\rho) \mu_0(d\rho)$ for any continuous bounded function f on $\mathfrak{S}(\mathcal{H})$.

3 Special continuity bound for $I(A:B|C)$.

Our main technical tool is the following lemma proved by modification of the Alicki-Fannes-Winter method [1, 29].

Lemma 3. *Let ρ and σ be states in $\mathfrak{S}(\mathcal{H}_{ABCD})$ such that ⁹ $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$ and $g(\varepsilon) \doteq (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)$. Let $I(A:B|C)_\rho$ be the CMI defined in (11),(12).*

A) *If the states ρ_{AD} and σ_{AD} are supported by some d -dimensional subspace \mathcal{H}_d of \mathcal{H}_{AD} then $I(A:B|C)_\rho$ and $I(A:B|C)_\sigma$ are finite and*

$$|I(A:B|C)_\rho - I(A:B|C)_\sigma| \leq 2\varepsilon \log d + 2g(\varepsilon). \quad (29)$$

If ρ and σ are qc-states with respect to the decomposition $(AD)(BC)$ then the term $2\varepsilon \log d$ in (29) can be replaced by $\varepsilon \log d$.

B) *Let \mathcal{H}_* be a subspace of \mathcal{H}_{AD} containing the supports of ρ_{AD} and σ_{AD} . If $\varepsilon \leq \frac{1}{2}$ and $\text{Tr} H_* \rho_{AD}, \text{Tr} H_* \sigma_{AD} \leq E < +\infty$ for some positive operator H_* in \mathcal{H}_* satisfying condition (19) then $I(A:B|C)_\rho$ and $I(A:B|C)_\sigma$ are finite and*

$$|I(A:B|C)_\rho - I(A:B|C)_\sigma| \leq 2\sqrt{2\varepsilon} F_{H_*}(E/\varepsilon) + 2g(\sqrt{2\varepsilon}), \quad (30)$$

where $F_{H_}(E) \doteq \sup_{\text{Tr} H_* \rho \leq E} H(\rho)$ is the function determined by formula (17).*

If ρ and σ are pure states then (30) holds with ε replaced by $\varepsilon^2/2$.

Proof. A) Following [29] introduce the state $\omega^* = (1 + \varepsilon)^{-1}(\rho + [\sigma - \rho]_+)$ in $\mathfrak{S}(\mathcal{H}_{ABCD})$. Then

$$\frac{1}{1 + \varepsilon} \rho + \frac{\varepsilon}{1 + \varepsilon} \tau_- = \omega^* = \frac{1}{1 + \varepsilon} \sigma + \frac{\varepsilon}{1 + \varepsilon} \tau_+,$$

where $\tau_+ = \varepsilon^{-1}[\rho - \sigma]_+$ and $\tau_- = \varepsilon^{-1}[\rho - \sigma]_-$ are states in $\mathfrak{S}(\mathcal{H}_{ABCD})$. By taking partial trace we obtain

$$\frac{1}{1 + \varepsilon} \rho_{ABC} + \frac{\varepsilon}{1 + \varepsilon} [\tau_-]_{ABC} = \omega_{ABC}^* = \frac{1}{1 + \varepsilon} \sigma_{ABC} + \frac{\varepsilon}{1 + \varepsilon} [\tau_+]_{ABC}.$$

Basic properties of the conditional mutual information and upper bound (8) imply

$$I(A:B|C)_\omega \leq I(A:BC)_\omega \leq I(AD:BC)_\omega \leq 2H(\omega_{AD}) \quad (31)$$

⁹Here and in all the "corollaries" of Lemma 3B we assume that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$ (instead of $\frac{1}{2}\|\rho - \sigma\|_1 = \varepsilon$), since we can not guarantee, in general, that the right hand side of (30) is a nondecreasing function of ε inspite of its tending to zero as $\varepsilon \rightarrow 0$ by condition (19).

for any state ω in $\mathfrak{S}(\mathcal{H}_{ABCD})$. Since the operators $\text{Tr}_{BC}[\rho - \sigma]_{\pm}$ are supported by the subspace \mathcal{H}_d , it follows from (31) that

$$I(A:B|C)_{\omega} \leq 2 \log d < +\infty, \quad \omega = \rho, \sigma, \tau_+, \tau_-. \quad (32)$$

By applying (15) to the above convex decompositions of ω_{ABC}^* we obtain

$$(1-p) [I(A:B|C)_{\rho} - I(A:B|C)_{\sigma}] \leq p [I(A:B|C)_{\tau_+} - I(A:B|C)_{\tau_-}] + 2h_2(p)$$

and

$$(1-p) [I(A:B|C)_{\sigma} - I(A:B|C)_{\rho}] \leq p [I(A:B|C)_{\tau_-} - I(A:B|C)_{\tau_+}] + 2h_2(p),$$

where $p = \frac{\varepsilon}{1+\varepsilon}$. These inequalities, upper bound (32) and nonnegativity of $I(A:B|C)$ imply (29).

If ρ and σ are qc -states with respect to the decomposition $(AD)(BC)$ then the above states τ_+ and τ_- are qc -states as well. So, by using (9) instead of (8) we obtain $\log d$ instead of $2 \log d$ in (32).

B) We may consider $I(A:B|C)$ as a function on $\mathfrak{S}(\mathcal{H}_{BC} \otimes \mathcal{H}_*)$. The assertion can be proved by applying Proposition 1 in [23] to this function by using the nonnegativity of $I(A:B|C)$, upper bound (31) and inequality (15). \square

4 Continuity bounds for output conditional mutual information

The quantum mutual information and its conditional version play basic role in analysis of informational properties of quantum systems and channels (see Sections 5-7). In this section we explore continuity properties of the conditional mutual information (defined in (11),(12)) at output of a channel acting on one subsystem of a tripartite system, i.e. the quantity $I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$, where $\Phi : A \rightarrow B$ is an arbitrary channel, C, D are any systems and ρ is a state in $\mathfrak{S}(\mathcal{H}_{ADC})$. We will obtain continuity bounds for the function

$$(\Phi, \rho) \mapsto I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$$

assuming that the set of all channels from A to B is equipped with the Bures distance (equivalent to the metric induced by the diamond norm, see Section 2.1). We will also obtain continuity bounds for the function

$$\Phi \mapsto I(B^n:D|C)_{\Phi^{\otimes n} \otimes \text{Id}_{CD}(\rho)}$$

for arbitrary $\rho \in \mathfrak{S}(\mathcal{H}_{A^nCD})$ and any natural n .

4.1 The function $(\Phi, \rho) \mapsto I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$

4.1.1 Finite input dimension

Proposition 1. *Let Φ and Ψ be quantum channels from finite-dimensional system A to arbitrary system B, C and D be any systems. Then*

$$|I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)} - I(B:D|C)_{\Psi \otimes \text{Id}_{CD}(\sigma)}| \leq 2\varepsilon \log d_A + 2\varepsilon \log 2 + 2g(\varepsilon), \quad (33)$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{ACD})$, where $\varepsilon = \frac{1}{2}\|\rho - \sigma\|_1 + \beta(\Phi, \Psi)$, $d_A \doteq \dim \mathcal{H}_A$ and $g(\varepsilon) = (1 + \varepsilon)h_2(\frac{\varepsilon}{1+\varepsilon})$.

If $\Phi = \Psi$ then the summand $2\varepsilon \log 2$ in (33) can be removed.

Continuity bound (33) is tight in both cases $\Phi = \Psi$ and $\rho = \sigma$ even for trivial C when $I(B:D|C) = I(B:D)$. The Bures distance $\beta(\Phi, \Psi)$ in (33) can be replaced by $\|\Phi - \Psi\|_\diamond^{1/2}$.

Proof. Let E be a common environment for the channels Φ and Ψ , so that Stinespring representations (5) hold with some isometries V_Φ and V_Ψ from \mathcal{H}_A into \mathcal{H}_{BE} . By Theorem 1 in [12] we may assume that $\|V_\Phi - V_\Psi\| = \beta(\Phi, \Psi)$. Then $\hat{\rho} = V_\Phi \otimes I_{CD} \rho V_\Phi^* \otimes I_{CD}$ and $\hat{\sigma} = V_\Psi \otimes I_{CD} \sigma V_\Psi^* \otimes I_{CD}$ are extensions of the states $\Phi \otimes \text{Id}_{CD}(\rho)$ and $\Psi \otimes \text{Id}_{CD}(\sigma)$. Lemma 1 implies

$$\|\hat{\rho} - \hat{\sigma}\|_1 \leq \|\rho - \sigma\|_1 + 2\|V_\Phi - V_\Psi\|. \quad (34)$$

Since the states $\hat{\rho}_{BE} = V_\Phi \rho_A V_\Phi^*$ and $\hat{\sigma}_{BE} = V_\Psi \sigma_A V_\Psi^*$ are supported by the subspace $V_\Phi \mathcal{H}_A \vee V_\Psi \mathcal{H}_A$ of \mathcal{H}_{BE} having dimension $\leq 2d_A$, Lemma 3A and inequality (34) imply (33). If $\Phi = \Psi$ then the above states $\hat{\rho}_{BE}$ and $\hat{\sigma}_{BE}$ are supported by the d_A -dimensional subspace $V_\Phi \mathcal{H}_A = V_\Psi \mathcal{H}_A$.

The tightness of continuity bound (33) in the case $\Phi = \Psi$ follows from Corollary 1 in [22].

The tightness of continuity bound (33) in the case $\rho = \sigma$ follows from the tightness of continuity bound (62) for the quantum capacity in Section 6. It can be directly shown by using the erasure channels $\Phi_{1/2}$ and $\Phi_{1/2-x}$ (see the proof of Theorem 1).

The last assertion of the proposition follows from the right inequality in (6) and monotonicity of the function $g(x)$. \square

4.1.2 Finite input energy

Continuity bound for the function $(\Phi, \rho) \mapsto I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$ under the energy constraint on ρ_A can be obtained by combining continuity bounds for

the functions $\rho \mapsto I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$ and $\Phi \mapsto I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$ under this constraint *not depending* on Φ and on ρ .

Proposition 2. *Let $\Phi : A \rightarrow B$ be an arbitrary quantum channel, C and D be any systems. If the Hamiltonian H_A of input system A satisfies condition (19) then the function $\rho_{ACD} \mapsto I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$ is uniformly continuous on the set of states with bounded energy of ρ_A and*

$$|I(B:D|C)_{\Phi \otimes \text{Id}_C(\rho)} - I(B:D|C)_{\Phi \otimes \text{Id}_C(\sigma)}| \leq 2\sqrt{2\varepsilon}F_{H_A}(E/\varepsilon) + 2g(\sqrt{2\varepsilon}) \quad (35)$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{ACD})$ such that $\text{Tr}H_A\rho_A, \text{Tr}H_A\sigma_A \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq \frac{1}{2}$, where $F_{H_A}(E) \doteq \sup_{\text{Tr}H_A\rho \leq E} H(\rho)$ is the function determined by formula (17).

If ρ and σ are pure states then (35) holds with ε replaced by $\varepsilon^2/2$.

If A is the ℓ -mode quantum oscillator then the function F_{H_A} in (35) can be replaced by its upper bound $\hat{F}_{\ell,\omega}$ defined in (20).

The main term in (35) tends to zero as $\varepsilon \rightarrow 0$, since condition (19) implies $F_{H_A}(E) = o(\sqrt{E})$ as $E \rightarrow +\infty$ (see Section 2.2.1).

Proof. Assume that the channel Φ has the Stinespring representation (1). Continuity bound (35) can be obtained from Lemma 3B by identifying \mathcal{H}_A and H_A with the subspace $V\mathcal{H}_A$ of \mathcal{H}_{BE} and the operator VH_AV^* in $V\mathcal{H}_A$ correspondingly. \square

The continuity bound for the function $\Phi \mapsto I(B:D|C)_{\Phi \otimes \text{Id}_{CD}(\rho)}$ under the energy constraint on ρ_A not depending on Φ and on ρ is presented in Proposition 4 below (the case $n = 1$).

4.2 The function $\Phi \mapsto I(B^n:D|C)_{\Phi^{\otimes n} \otimes \text{Id}_{CD}(\rho)}$

4.2.1 Finite input dimension

The following proposition is a d_A -version of Proposition 3A in [22] proved by using the Leung-Smith telescopic trick from [13].

Proposition 3. *Let Φ and Ψ be quantum channels from finite-dimensional system A to arbitrary system B , C and D be any systems and $n \in \mathbb{N}$. Then*

$$|I(B^n:D|C)_{\Phi^{\otimes n} \otimes \text{Id}_{CD}(\rho)} - I(B^n:D|C)_{\Psi^{\otimes n} \otimes \text{Id}_{CD}(\rho)}| \leq 2n(\varepsilon \log(2d_A) + g(\varepsilon))$$

for any state ρ in $\mathfrak{S}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{CD})$, where $\varepsilon = \beta(\Phi, \Psi)$ and $d_A = \dim \mathcal{H}_A$.

This continuity bound is tight even for trivial C (for each given n and large d_A). The Bures distance $\beta(\Phi, \Psi)$ in it can be replaced by $\|\Phi - \Psi\|_\diamond^{1/2}$.

Proof. Let E be a common environment for the channels Φ and Ψ , so that Stinespring representations (5) hold with some isometries V_Φ and V_Ψ from \mathcal{H}_A into \mathcal{H}_{BE} . By Theorem 1 in [12] we may assume that $\|V_\Phi - V_\Psi\| = \beta(\Phi, \Psi)$.

Consider the states

$$\sigma_k = \Phi^{\otimes k} \otimes \Psi^{\otimes (n-k)} \otimes \text{Id}_{CD}(\rho), \quad k = 0, 1, \dots, n.$$

We have

$$\begin{aligned} |I(B^n : D|C)_{\sigma_n} - I(B^n : D|C)_{\sigma_0}| &= \left| \sum_{k=1}^n I(B^n : D|C)_{\sigma_k} - I(B^n : D|C)_{\sigma_{k-1}} \right| \\ &\leq \sum_{k=1}^n |I(B^n : D|C)_{\sigma_k} - I(B^n : D|C)_{\sigma_{k-1}}|. \end{aligned} \quad (36)$$

By using the chain rule (13) we obtain for each k

$$\begin{aligned} I(B^n : D|C)_{\sigma_k} - I(B^n : D|C)_{\sigma_{k-1}} &= I(B_1 \dots B_{k-1} B_{k+1} \dots B_n : D|C)_{\sigma_k} \\ &\quad + I(B_k : D|B_1 \dots B_{k-1} B_{k+1} \dots B_n C)_{\sigma_k} \\ &\quad - I(B_1 \dots B_{k-1} B_{k+1} \dots B_n : D|C)_{\sigma_{k-1}} \\ &\quad - I(B_k : D|B_1 \dots B_{k-1} B_{k+1} \dots B_n C)_{\sigma_{k-1}} \\ &= I(B_k : D|B_1 \dots B_{k-1} B_{k+1} \dots B_n C)_{\sigma_k} \\ &\quad - I(B_k : D|B_1 \dots B_{k-1} B_{k+1} \dots B_n C)_{\sigma_{k-1}}, \end{aligned} \quad (37)$$

where it was used that $\text{Tr}_{B_k} \sigma_k = \text{Tr}_{B_k} \sigma_{k-1}$. Note that the finite entropy of the states $\rho_{A_1}, \dots, \rho_{A_n}$, upper bound (14) and monotonicity of the conditional mutual information under local channels guarantee finiteness of all the terms in (36) and (37).

To estimate the last difference in (37) consider the states

$$\hat{\sigma}_k = W_k \otimes V_\Phi^k \otimes I_{CD} \rho W_k^* \otimes [V_\Phi^k]^* \otimes I_{CD} \quad (38)$$

and

$$\hat{\sigma}_{k-1} = W_k \otimes V_\Psi^k \otimes I_{CD} \rho W_k^* \otimes [V_\Psi^k]^* \otimes I_{CD} \quad (39)$$

in $\mathfrak{S}(\mathcal{H}_{B^n E^n C D})$, where $W_k = V_\Phi^1 \otimes \dots \otimes V_\Phi^{k-1} \otimes V_\Psi^{k+1} \otimes \dots \otimes V_\Psi^n$ is an isometry from $\mathcal{H}_{A^n \setminus A_k}$ into $\mathcal{H}_{[BE]^n \setminus [BE]_k}$, $V_\Phi^k \cong V_\Phi$ and $V_\Psi^k \cong V_\Psi$ are isometries from \mathcal{H}_{A_k} into $\mathcal{H}_{B_k E_k}$. It follows from (5) that these states are extensions of the states σ_k and σ_{k-1} , i.e. $\text{Tr}_{E^n} \hat{\sigma}_k = \sigma_k$ and $\text{Tr}_{E^n} \hat{\sigma}_{k-1} = \sigma_{k-1}$. Since

$$[\hat{\sigma}_k]_{B_k E_k} = V_\Phi^k \rho_{A_k} [V_\Phi^k]^* \quad \text{and} \quad [\hat{\sigma}_{k-1}]_{B_k E_k} = V_\Psi^k \rho_{A_k} [V_\Psi^k]^*,$$

Lemma 3A implies

$$|I(B_k : D | X)_{\sigma_k} - I(B_k : D | X)_{\sigma_{k-1}}| \leq 2\varepsilon' \log(2d_A) + 2g(\varepsilon'), \quad (40)$$

where $X = B_1 \dots B_{k-1} B_{k+1} \dots B_n C$ and $\varepsilon' = \frac{1}{2} \|\hat{\sigma}_k - \hat{\sigma}_{k-1}\|_1$. By using Lemma 1 we obtain

$$\varepsilon' \leq \|W_k \otimes V_\Phi^k \otimes I_{CD} - W_k \otimes V_\Psi^k \otimes I_{CD}\| = \|V_\Phi - V_\Psi\| = \beta(\Phi, \Psi) = \varepsilon.$$

Hence, it follows from (37) and (40) that

$$|I(B^n : D | C)_{\sigma_k} - I(B^n : D | C)_{\sigma_{k-1}}| \leq 2\varepsilon \log(2d_A) + 2g(\varepsilon).$$

This and (36) imply the required inequality (since $\Phi^{\otimes n} \otimes \text{Id}_{CD}(\rho) = \sigma_n$ and $\Psi^{\otimes n} \otimes \text{Id}_{CD}(\rho) = \sigma_0$).

The tightness of the continuity bound in Proposition 3 follows from the tightness of continuity bound (33) in the case $\rho = \sigma$, since for arbitrary channel $\Phi : A \rightarrow B$, any system D and a state $\rho \in \mathfrak{S}(\mathcal{H}_{AD})$ we have

$$I(B^n : D^n)_{\Phi^{\otimes n} \otimes \text{Id}_{D^n}(\rho^{\otimes n})} = nI(B : D)_{\Phi \otimes \text{Id}_D(\rho)}.$$

The last assertion of the proposition follows from the right inequality in (6) and monotonicity of the function $g(x)$. \square

4.2.2 Finite input energy

In this subsection we assume that H_A is the Hamiltonian of system A satisfying condition (19), $F_{H_A}(E) \doteq \sup_{\text{Tr} H_A \rho \leq E} H(\rho)$ is the increasing concave function on $[E_0, +\infty)$ determined by formula (17) and $\{\gamma(d) = F_{H_A}^{-1}(\log d)\}_{d \geq d_0}$ is the sequence tending to $+\infty$ as $d \rightarrow +\infty$ introduced in Lemma 2, where E_0 is the minimal eigenvalue of H_A and d_0 is its multiplicity (see Section 2.2.1).

Proposition 4. *Let $\Phi : A \rightarrow B$ and $\Psi : A \rightarrow B$ be arbitrary quantum channels, C and D be any systems and $n \in \mathbb{N}$. Then*

$$|I(B^n : D|C)_{\Phi^{\otimes n} \otimes \text{Id}_{CD}(\rho)} - I(B^n : D|C)_{\Psi^{\otimes n} \otimes \text{Id}_{CD}(\rho)}| \leq n(C(\varepsilon, E) + 2g(\varepsilon)) \quad (41)$$

for any state ρ in $\mathfrak{S}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{CD})$ such that $\text{Tr} H_A \rho_{A_k} \leq E_k < +\infty$ for $k = \overline{1, n}$, where $\varepsilon = \beta(\Phi, \Psi) \leq \|\Phi - \Psi\|_{\diamond}^{1/2}$,

$$C(\varepsilon, E) \doteq \min_{d \geq d_0} \left[4\delta(d) \sqrt[4]{E} F_{H_A}(\sqrt{\gamma(d)E/2}) + 4g(\delta(d) \sqrt[4]{E}) + 2\varepsilon \log(2d) \right] \quad (42)$$

(the minimum is over natural numbers), $E = \frac{1}{n} \sum_{k=1}^n E_k$ and $\delta(d) = \sqrt[4]{8/\gamma(d)}$.

If ρ is a pure state then $C(\varepsilon, E)$ in (41) can be replaced by

$$C_*(\varepsilon, E) \doteq \min_{d \geq d_0} \left[\left(4\sqrt{2E/\gamma(d)} + 2\varepsilon \right) \log d + 4g(\sqrt{2E/\gamma(d)}) \right] + 2\varepsilon \log 2. \quad (43)$$

For given E the quantities $C(\varepsilon, E)$ and $C_*(\varepsilon, E)$ tend to zero as $\varepsilon \rightarrow +0$.

Remark 2. The quantities $C(\varepsilon, E)$ and $C_*(\varepsilon, E) < C(\varepsilon, E)$ are determined by the function F_{H_A} , i.e by the Hamiltonian H_A of system A . They can be calculated for any $\varepsilon > 0$ and $E > E_0$ by using explicit expression for F_{H_A} . If this expression is not known (or too complicated) we can use any upper bound \widehat{F} of F_{H_A} provided that \widehat{F} is a concave nonnegative function such that $\widehat{F}(E) = o(\sqrt{E})$ as $E \rightarrow +\infty$. This follows from the proofs of Proposition 4 and of Lemma 2. It means that we can replace the quantities $C(\varepsilon, E)$ and $C_*(\varepsilon, E)$ in Proposition 4 by the analogous quantities determined by formulae (42) and (43) with F_{H_A} , $\gamma(d)$ and d_0 replaced by \widehat{F} , $\hat{\gamma}(d) \doteq \widehat{F}^{-1}(\log d)$ and $e^{\widehat{F}(0)}$ correspondingly. By using this approach one can obtain the following specification of Proposition 4:

If A is the ℓ -mode quantum oscillator with the frequencies $\omega_1, \dots, \omega_\ell$ and $\varepsilon \leq 1$ then the quantities $C(\varepsilon, E)$ and $C_*(\varepsilon, E)$ in Proposition 4 can be replaced, respectively, by

$$\widehat{C}(\varepsilon, E) = 6\varepsilon\ell \log \left[\frac{12E}{\ell E_*} \right] + 16\ell\eta(\varepsilon) + 4g(\varepsilon) + 2\varepsilon \log 2 + 2\varepsilon r(\varepsilon^2) \quad (44)$$

and

$$\widehat{C}_*(\varepsilon, E) = 4\varepsilon\ell \log \left[\frac{25E}{\ell E_*} \right] + 8\ell\eta(\varepsilon) + 4g(\varepsilon/2) + 2\varepsilon \log 2 + 4\varepsilon r(\varepsilon), \quad (45)$$

where $E_* = \left[\prod_{i=1}^{\ell} \hbar \omega_i \right]^{1/\ell}$, $\eta(x) = -x \log x$ and $r(x) = \frac{x^{2\ell}}{(8+x^2)^\ell}$.

This assertion is proved by using the function $\hat{F}_{\ell,\omega}$ and the corresponding sequence $\hat{\gamma}(d)$ defined, respectively, in (20) and (21). To obtain (44) one should make elementary estimation the quantity

$$4\sqrt[4]{8E/\hat{\gamma}(d)} \hat{F}_{\ell,\omega}(\sqrt{\hat{\gamma}(d)E/2}) + 4g(\sqrt[4]{8E/\hat{\gamma}(d)}) + 2\varepsilon \log(2d),$$

where d is the minimal natural number such that $\sqrt[4]{8E/\hat{\gamma}(d)} \leq \varepsilon$. To obtain (45) it suffices to estimate the quantity

$$(4\sqrt{2E/\hat{\gamma}(d)} + 2\varepsilon) \log d + 4g(\sqrt{2E/\hat{\gamma}(d)}) + 2\varepsilon \log 2,$$

where d is the minimal natural number such that $2\sqrt{2E/\hat{\gamma}(d)} \leq \varepsilon$.

Proof of Proposition 4. We can repeat the arguments from the proof of Proposition 3 up to the estimation of the last difference in (37).

Let $d \geq d_0$ and $\Delta_d^k = \sqrt{2E_k/\gamma(d)}$. For given k Lemma 2B implies existence of a state ϱ_k in $\mathfrak{S}(\mathcal{H}_{A^n C D})$ such that

$$\text{rank}[\varrho_k]_{A_k} \leq d, \quad \text{Tr} H_A[\varrho_k]_{A_k} \leq E_k \quad \text{and} \quad \frac{1}{2} \|\rho - \varrho_k\|_1 \leq \Delta_d^k. \quad (46)$$

Let $\hat{\varsigma}_k$ and $\hat{\varsigma}_{k-1}$ be the states in $\mathfrak{S}(\mathcal{H}_{B^n E^n C D})$ defined respectively by formulas (38) and (39) with ρ replaced by ϱ_k . Then, by repeating the arguments after these formulas based on Lemma 3A we obtain

$$|I(B_k : D|X)_{\hat{\varsigma}_k} - I(B_k : D|X)_{\hat{\varsigma}_{k-1}}| \leq 2\varepsilon \log(2d) + 2g(\varepsilon),$$

where $\varepsilon = \beta(\Phi, \Psi)$.

Since $\|\hat{\sigma}_{k-1} - \hat{\varsigma}_{k-1}\|_1 = \|\hat{\sigma}_k - \hat{\varsigma}_k\|_1 = \|\rho - \varrho_k\|_1$ (where $\hat{\sigma}_k$ and $\hat{\sigma}_{k-1}$ are the states defined respectively by formulas (38) and (39)) and $\text{Tr} H_A \rho_{A_k} \leq E_k$, by using Lemma 3B and (46) one can show that

$$|I(B_k : D|X)_{\sigma_{k-1}} - I(B_k : D|X)_{\hat{\varsigma}_{k-1}}| \quad \text{and} \quad |I(B_k : D|X)_{\sigma_k} - I(B_k : D|X)_{\hat{\varsigma}_k}|$$

do not exceed the quantity

$$\begin{aligned} & 2\sqrt{2\Delta_d^k} F_{H_A}(E_k/\Delta_d^k) + 2g(\sqrt{2\Delta_d^k}) \\ & = 2\delta(d) \sqrt[4]{E_k} F_{H_A}(\sqrt{\gamma(d)E_k/2}) + 2g(\delta(d) \sqrt[4]{E_k}). \end{aligned} \quad (47)$$

Thus, the last difference in (37) is upper bounded by

$$B_k(d) \doteq 4\delta(d)\sqrt[4]{E_k} F_{H_A}\left(\sqrt{\gamma(d)E_k/2}\right) + 4g\left(\delta(d)\sqrt[4]{E_k}\right) + 2\varepsilon \log(2d) + 2g(\varepsilon).$$

Hence, it follows from (36) that the left hand side of (41) do not exceed the sum $\sum_{k=1}^n B_k(d)$. By using the concavity¹⁰ of the functions $\sqrt[4]{x} F_{H_A}(\sqrt{x})$, $\sqrt[4]{x}$ and $g(x)$ along with the monotonicity of $g(x)$, it is easy to show that $\frac{1}{n} \sum_{k=1}^n B_k(d)$ is upper bounded by

$$4\delta(d)\sqrt[4]{E} F_{H_A}\left(\sqrt{\gamma(d)E/2}\right) + 4g\left(\delta(d)\sqrt[4]{E}\right) + 2\varepsilon \log(2d) + 2g(\varepsilon).$$

This implies (41).

If the state ρ is pure then the above state ϱ_k can be taken pure for each k (by Lemma 2B). So, by the last assertion of Lemma 3B, the upper bound (47) can be replaced by

$$2\Delta_d^k F_{H_A}(2E_k/(\Delta_d^k)^2) + 2g(\Delta_d^k) = 2\sqrt{2E_k/\gamma(d)} \log d + 2g\left(\sqrt{2E_k/\gamma(d)}\right),$$

where the obvious equality $F_{H_A}(\gamma(d)) = \log d$ was used. So, by using concavity of the functions \sqrt{x} and $g(x)$ along with the monotonicity of $g(x)$ we obtain that the left hand side of (41) is upper bounded by

$$\left(4\sqrt{2E/\gamma(d)} + 2\varepsilon\right) \log d + 4g\left(\sqrt{2E/\gamma(d)}\right) + 2\varepsilon \log 2 + 2g(\varepsilon).$$

The last assertion of the proposition can be easily proved by noting that $F_{H_A}(E) = o(\sqrt{E})$ as $E \rightarrow +\infty$ by condition (19). \square

5 Continuity bounds for the output Holevo quantity

In analysis of information properties of quantum channels we have to consider the output Holevo quantity of a given channel $\Phi : A \rightarrow B$ corresponding to a discrete or continuous ensemble μ of input quantum states, i.e. the quantity

$$\chi(\Phi(\mu)) = \int H(\Phi(\rho) \parallel \Phi(\bar{\rho}(\mu))) \mu(d\rho) = H(\Phi(\bar{\rho}(\mu))) - \int H(\Phi(\rho)) \mu(d\rho),$$

¹⁰The concavity of the function $\sqrt[4]{x} F_{H_A}(\sqrt{x})$ follows from the concavity and nonnegativity of the function $F_{H_A}(x)$. This can be shown by calculation of the second derivative.

where the second formula is valid under the condition $H(\Phi(\bar{\rho}(\mu))) < +\infty$.

We will consider the output Holevo quantity $\chi(\Phi(\mu))$ as a function of a pair (channel Φ , input ensemble μ) assuming that

- the set of discrete ensembles is equipped with one of the metrics D_0 , D_* and D_K described in Section 2.3.1;
- the set of generalized (continuous) ensembles is equipped with the Kantorovich metric D_K defined in (28);
- the set of all channels is equipped with the Bures distance β defined in (4) and equivalent to the metric induced by the diamond norm (3).

5.1 Finite input dimension

Speaking about the output Holevo quantity $\chi(\Phi(\mu))$ of a channel Φ with finite-dimensional input space we restrict attention to discrete ensembles μ , i.e. $\mu = \{p_i, \rho_i\}$, for which

$$\chi(\Phi(\mu)) = \chi(\{p_i, \Phi(\rho_i)\}) \doteq \sum_i p_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})), \quad \bar{\rho} = \sum_i p_i \rho_i.$$

Tight continuity bound for the function $(\Phi, \mu) \mapsto \chi(\Phi(\mu))$ depending on the input dimension of a channel is presented in the following proposition.

Proposition 5. *Let Φ and Ψ be quantum channels from a finite-dimensional system A to arbitrary system B . Let μ and ν be discrete ensembles of states in $\mathfrak{S}(\mathcal{H}_A)$. Then*

$$|\chi(\Phi(\mu)) - \chi(\Psi(\nu))| \leq \varepsilon \log d_A + \varepsilon \log 2 + 2g(\varepsilon), \quad (48)$$

where $d_A \doteq \dim \mathcal{H}_A$, $\varepsilon = D_*(\mu, \nu) + \beta(\Phi, \Psi)$ and $g(\varepsilon) = (1 + \varepsilon)h_2(\frac{\varepsilon}{1+\varepsilon})$.

If $\Phi = \Psi$ then the summand $\varepsilon \log 2$ in (48) can be removed.

Continuity bound (48) is tight in both cases $\Phi = \Psi$ and $\mu = \nu$. The metric D_* in (48) can be replaced by any of the metrics D_0 and D_K , the Bures distance $\beta(\Phi, \Psi)$ can be replaced by $\|\Phi - \Psi\|_\diamond^{1/2}$.

Proof. Assume that $\mu = \{p_i, \rho_i\}$ and $\nu = \{q_i, \sigma_i\}$. Take any $\epsilon > 0$. Let $\{\tilde{p}_i, \tilde{\rho}_i\}$ and $\{\tilde{q}_i, \tilde{\sigma}_i\}$ be ensembles belonging respectively to the sets $\mathcal{E}(\{p_i, \rho_i\})$ and $\mathcal{E}(\{q_i, \sigma_i\})$ such that

$$D_*(\{p_i, \rho_i\}, \{q_i, \sigma_i\}) \geq D_0(\{\tilde{p}_i, \tilde{\rho}_i\}, \{\tilde{q}_i, \tilde{\sigma}_i\}) - \epsilon \quad (49)$$

(see the definition (23) of D_*). Let E be a common environment for the channels Φ and Ψ such that representation (5) holds with the isometries V_Φ and V_Ψ from \mathcal{H}_A into \mathcal{H}_{BE} for which $\|V_\Phi - V_\Psi\| = \beta(\Phi, \Psi)$ [12, Th.1].

Consider the qc -states

$$\hat{\rho} = \sum_i \tilde{p}_i V_\Phi \tilde{\rho}_i V_\Phi^* \otimes |i\rangle\langle i| \quad \text{and} \quad \hat{\sigma} = \sum_i \tilde{q}_i V_\Psi \tilde{\sigma}_i V_\Psi^* \otimes |i\rangle\langle i|$$

in $\mathfrak{S}(\mathcal{H}_{BEC})$, where $\{|i\rangle\}$ is a basic in \mathcal{H}_C . Representation (22) implies

$$\chi(\{p_i, \Phi(\rho_i)\}) = \chi(\{\tilde{p}_i, \Phi(\tilde{\rho}_i)\}) = I(B:C)_{\hat{\rho}}$$

and

$$\chi(\{q_i, \Psi(\sigma_i)\}) = \chi(\{\tilde{q}_i, \Psi(\tilde{\sigma}_i)\}) = I(B:C)_{\hat{\sigma}},$$

where the first equalities follow from the obvious observation:

$$\tilde{\mu} \in \mathcal{E}(\mu) \quad \Rightarrow \quad \Phi(\tilde{\mu}) \in \mathcal{E}(\Phi(\mu)) \quad \Rightarrow \quad \chi(\Phi(\tilde{\mu})) = \chi(\Phi(\mu))$$

valid for any ensemble μ and any channel Φ .

Lemma 1 implies

$$\|\hat{\rho} - \hat{\sigma}\|_1 \leq \sum_i \|\tilde{p}_i \tilde{\rho}_i - \tilde{q}_i \tilde{\sigma}_i\|_1 + 2\|V_\Phi - V_\Psi\|. \quad (50)$$

Since the states $\hat{\rho}_{BE}$ and $\hat{\sigma}_{BE}$ are supported by the subspace $V_\Phi \mathcal{H}_A \vee V_\Psi \mathcal{H}_A$ of \mathcal{H}_{BE} having dimension $\leq 2d_A$, Lemma 3A and (49)-(50) imply (48). If $\Phi = \Psi$ then the above states $\hat{\rho}_{BE}$ and $\hat{\sigma}_{BE}$ are supported by the d_A -dimensional subspace $V_\Phi \mathcal{H}_A = V_\Psi \mathcal{H}_A$.

The tightness of continuity bound (48) in the case $\Phi = \Psi$ follows from the tightness of the continuity bound (27), see Proposition 5 in [22].

The tightness of continuity bound (48) in the case $\{p_i, \rho_i\} = \{q_i, \sigma_i\}$ follows from the tightness of continuity bound (60) for the Holevo capacity in Section 6 (which is derived from (48)). It can be directly shown by using the erasure channels $\Phi_{1/2}$ and $\Phi_{1/2-x}$ (see the proof of Theorem 1).

Since the function $g(x)$ is increasing, the last assertion of the proposition follows from inequalities (24),(26) and the right inequality in (6). \square

5.2 Finite input energy

Speaking about the output Holevo quantity $\chi(\Phi(\mu))$ of a channel Φ between infinite-dimensional quantum systems A and B we will assume that μ runs over the set of all generalized (continuous) ensembles $\mathcal{P}(\mathcal{H}_A)$ equipped with the weak convergence topology (see Section 2.3.2). Specifications concerning the case of discrete ensembles will be given as additional remarks.

We will analyse the function $(\Phi, \mu) \mapsto \chi(\Phi(\mu))$ under the constraint on the average energy of μ , i.e. under the condition

$$E(\mu) \doteq \text{Tr} H_A \bar{\rho}(\mu) = \int \text{Tr} H_A \rho \mu(d\rho) \leq E,$$

where H_A is the Hamiltonian of input system A and $E > E_0 \doteq \inf_{\|\varphi\|=1} \langle \varphi | H_A | \varphi \rangle$.

Continuity bound for the function $(\Phi, \mu) \mapsto \chi(\Phi(\mu))$ under the constraint on the average energy of μ can be obtained by combining continuity bounds for the functions $\mu \mapsto \chi(\Phi(\mu))$ and $\Phi \mapsto \chi(\Phi(\mu))$ *not depending* on Φ and on μ presented in the following two propositions.

Proposition 6. *Let $\Phi : A \rightarrow B$ be an arbitrary quantum channel. If the Hamiltonian H_A of input system A satisfies condition (19) then the function $\mu \mapsto \chi(\Phi(\mu))$ is uniformly continuous on the subset of $\mathcal{P}(\mathcal{H}_A)$ consisting of ensembles μ with bounded average energy $E(\mu) \doteq \text{Tr} H_A \bar{\rho}(\mu)$ and*

$$|\chi(\Phi(\mu)) - \chi(\Phi(\nu))| \leq 2\sqrt{2\varepsilon} F_{H_A}(E/\varepsilon) + 2g(\sqrt{2\varepsilon}) \quad (51)$$

for any ensembles μ and ν such that $E(\mu), E(\nu) \leq E$ and $D_K(\mu, \nu) \leq \varepsilon \leq \frac{1}{2}$, where $F_{H_A}(E) \doteq \sup_{\text{Tr} H_A \rho \leq E} H(\rho)$ is the function determined by formula (17).

If μ and ν are discrete ensembles then the metric D_K can be replaced by any of the metrics D_0 and D_* .

If A is the ℓ -mode quantum oscillator then the function F_{H_A} in (51) can be replaced by its upper bound $\hat{F}_{\ell, \omega}$ defined in (20).

Proof. For discrete ensembles μ and ν the inequality (51) with ε in $[D_*(\mu, \nu), 1/2]$ is proved in [23, Pr.7]. It follows from (24) and (26) that this inequality holds for any $\varepsilon \in [D(\mu, \nu), 1/2]$, where D is either D_0 or D_K .

For arbitrary generalized ensembles μ and ν there exist sequences $\{\mu_n\}$ and $\{\nu_n\}$ of discrete ensembles converging respectively to μ and ν such that

$$\lim_{n \rightarrow \infty} \chi(\Phi(\mu_n)) = \chi(\Phi(\mu)), \quad \lim_{n \rightarrow \infty} \chi(\Phi(\nu_n)) = \chi(\Phi(\nu))$$

and $\bar{\rho}(\mu_n) = \bar{\rho}(\mu)$, $\bar{\rho}(\nu_n) = \bar{\rho}(\nu)$ for all n . Such sequences can be obtained by using the construction from the proof of Lemma 1 in [9] and taking into account the lower semicontinuity of the function $\mu \mapsto \chi(\Phi(\mu))$ [9, Pr.1]. Since inequality (51) holds for the ensembles μ_n and ν_n for all n and $D_K(\mu_n, \nu_n)$ tends to $D_K(\mu, \nu)$, the above limit relations imply the validity of (51) for the ensembles μ and ν . \square

Proposition 7. *Let μ be an arbitrary ensemble in $\mathcal{P}(\mathcal{H}_A)$ such that $E(\mu) \doteq \text{Tr} H_A \bar{\rho}(\mu) \leq E$. If the Hamiltonian H_A of input system A satisfies condition (19) then the function $\Phi \mapsto \chi(\Phi(\mu))$ is uniformly continuous on the set of all channels from A to any system B with respect to the Bures distance (the diamond norm) and*

$$|\chi(\Phi(\mu)) - \chi(\Psi(\mu))| \leq C(\varepsilon, E) + 2g(\varepsilon) \quad (52)$$

for any channels $\Phi : A \rightarrow B$ and $\Psi : A \rightarrow B$, where $\varepsilon = \beta(\Phi, \Psi) \leq \|\Phi - \Psi\|_\diamond^{1/2}$ and $C(\varepsilon, E)$ is the quantity defined in (42) tending to zero as $\varepsilon \rightarrow +0$.

If A is the ℓ -mode quantum oscillator and $\varepsilon \leq 1$ then the quantity $C(\varepsilon, E)$ in (52) can be replaced by its upper bound $\hat{C}(\varepsilon, E)$ defined in (44).

Proof. If μ is a discrete ensemble then the validity of (52) is derived from Proposition 4 with $n = 1$ and trivial C by using representation (22).

If μ is an arbitrary ensemble then the validity of (52) can be proved by approximation, since for any ensemble μ there exists a sequences $\{\mu_n\}$ of discrete ensembles weakly converging to μ such that

$$\bar{\rho}(\mu_n) = \bar{\rho}(\mu) \quad \text{for all } n \quad \text{and} \quad \lim_{n \rightarrow \infty} \chi(\Phi(\mu_n)) = \chi(\Phi(\mu))$$

for any channel Φ . Such a sequence can be obtained by using the construction from the proof of Lemma 1 in [9] and by taking into account the lower semicontinuity of the function $\mu \mapsto \chi(\Phi(\mu))$ [9, Pr.1].

The last assertion of the proposition follows from Remark 2. \square

6 Continuity bounds for basic capacities of channels with finite input dimension.

Continuity bounds for basic capacities of quantum channels with finite output dimension d_B are obtained by Leung and Smith in [13]. The main term in

all these bounds has the form $C\varepsilon \log d_B$ for some constant C , where ε is a distance between two channels (the diamond norm of their difference). These continuity bounds are essentially refined in [22] by using modification of the Leung-Smith approach (consisting in using the conditional mutual information instead of the conditional entropy).

In this section we consider quantum channels with finite input dimension¹¹ d_A and obtain continuity bounds for basic capacities of such channels with the main term $C\varepsilon \log d_A$, where ε is the Bures distance between quantum channels described in Section 2.1.

The *Holevo capacity* of a channel $\Phi : A \rightarrow B$ is defined as follows

$$\bar{C}(\Phi) = \sup_{\{p_i, \rho_i\}} \chi(\{p_i, \Phi(\rho_i)\}), \quad (53)$$

where the supremum is over all discrete ensembles of input states. This quantity determines the ultimate rate of transmission of classical information through a channel when nonentangled input encoding is used [7, 26].

By the Holevo-Schumacher-Westmoreland theorem the *classical capacity* of a channel $\Phi : A \rightarrow B$ is given by the regularized expression

$$C(\Phi) = \lim_{n \rightarrow +\infty} n^{-1} \bar{C}(\Phi^{\otimes n}). \quad (54)$$

The *classical entanglement-assisted capacity* of a quantum channel determines the ultimate rate of transmission of classical information when an entangled state between the input and the output of a channel is used as an additional resource (see details in [7, 26]). By the Bennett-Shor-Smolín-Thapalyal theorem the classical entanglement-assisted capacity of a channel $\Phi : A \rightarrow B$ is given by the expression

$$C_{\text{ea}}(\Phi) = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_A)} I(\Phi, \rho), \quad (55)$$

¹¹Channels with finite input dimension and infinite output dimension may appear as subchannels of "real" infinite-dimensional channels if we use for coding information only states supported by some finite-dimensional subspace of the input space. One can conjecture that speaking about energy-constrained capacities of infinite-dimensional channels from a given system to any other systems we may consider (permuting arbitrarily small error) that all these channels have *the same finite-dimensional input space* – the subspace corresponding to the minimal eigenvalues of the input Hamiltonian. For the Holevo capacity and the entanglement-assisted classical capacity this conjecture is proved in [23, Th.2].

in which $I(\Phi, \rho)$ is the quantum mutual information of a channel Φ at a state ρ defined as follows

$$I(\Phi, \rho) = I(B:R)_{\Phi \otimes \text{Id}_R(\hat{\rho})}, \quad (56)$$

where $\mathcal{H}_R \cong \mathcal{H}_A$ and $\hat{\rho}$ is a pure state in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\hat{\rho}_A = \rho$.

The *quantum capacity* of a channel characterizes the ultimate rate of transmission of quantum information (quantum states) through a channel (see details in [7, 26]). By the Lloyd-Devetak-Shor theorem the quantum capacity of a channel $\Phi : A \rightarrow B$ is given by the regularized expression

$$Q(\Phi) = \lim_{n \rightarrow +\infty} n^{-1} \bar{Q}(\Phi^{\otimes n}), \quad (57)$$

where $\bar{Q}(\Phi)$ is the maximal value of the coherent information $I_c(\Phi, \rho) \doteq H(\Phi(\rho)) - H(\hat{\Phi}(\rho))$ over all input states $\rho \in \mathfrak{S}(\mathcal{H}_A)$ (here $\hat{\Phi}$ is a complementary channel to the channel Φ defined in (2)).

The *private capacity* is the capacity of a channel for classical communication with the additional requirement that almost no information is sent to the environment (see details in [7, 26]). By the Devetak theorem the private capacity of a channel $\Phi : A \rightarrow B$ is given by the regularized expression

$$C_p(\Phi) = \lim_{n \rightarrow +\infty} n^{-1} \bar{C}_p(\Phi^{\otimes n}), \quad (58)$$

where

$$\bar{C}_p(\Phi) = \sup_{\{p_i, \rho_i\}} \left[\chi(\{p_i, \Phi(\rho_i)\}) - \chi(\{p_i, \hat{\Phi}(\rho_i)\}) \right] \quad (59)$$

(the supremum is over all discrete ensembles of input states and $\hat{\Phi}$ is the complementary channel to the channel Φ defined in (2)).

Now we consider continuity bounds for all the above capacities depending on the input dimension. For the entanglement-assisted classical capacity the tight continuity bound

$$|C_{\text{ea}}(\Phi) - C_{\text{ea}}(\Psi)| \leq 2\varepsilon \log d_A + 2g(\varepsilon),$$

where $\varepsilon = \frac{1}{2} \|\Phi - \Psi\|_{\diamond}$, is obtained in [22, Pr.8]. It is easy to show (by using formula (65) below with $n = 1$) that the same continuity bound holds for the quantity $\bar{Q}(\Phi)$.

For others basic capacities tight and close-to-tight continuity bounds depending on input dimension are presented in the following theorem, in which

the Bures distance $\beta(\Phi, \Psi)$ described in Section 2.1 is used as a measure of divergence between channels Φ and Ψ instead of $\frac{1}{2}\|\Phi - \Psi\|_\diamond$.

Theorem 1. *Let Φ and Ψ be quantum channels from finite-dimensional system A to arbitrary system B .¹² Then*

$$|\bar{C}(\Phi) - \bar{C}(\Psi)| \leq \varepsilon \log d_A + \varepsilon \log 2 + 2g(\varepsilon), \quad (60)$$

$$|C(\Phi) - C(\Psi)| \leq 2\varepsilon \log d_A + 2\varepsilon \log 2 + 2g(\varepsilon), \quad (61)$$

$$|Q(\Phi) - Q(\Psi)| \leq 2\varepsilon \log d_A + 2\varepsilon \log 2 + 2g(\varepsilon), \quad (62)$$

$$|\bar{C}_p(\Phi) - \bar{C}_p(\Psi)| \leq 2\varepsilon \log d_A + 2\varepsilon \log 2 + 2g(\varepsilon), \quad (63)$$

$$|C_p(\Phi) - C_p(\Psi)| \leq 4\varepsilon \log d_A + 4\varepsilon \log 2 + 4g(\varepsilon), \quad (64)$$

where $d_A \doteq \dim \mathcal{H}_A$, $\varepsilon = \beta(\Phi, \Psi)$ and $g(\varepsilon) = (1 + \varepsilon)h_2(\frac{\varepsilon}{1+\varepsilon})$.

The continuity bounds (60), (62) and (63) are tight. In all the inequalities (60)-(64) the Bures distance $\beta(\Phi, \Psi)$ can be replaced by $\|\Phi - \Psi\|_\diamond^{1/2}$.

Proof. Continuity bound (60) directly follows from Proposition 5 in Section 5.1 and the definition of the Holevo capacity.

Continuity bound (61) is obtained by using representation (22), Proposition 3 in Section 4.2 and Lemma 12 in [13].

To prove continuity bound (62) note that the coherent information can be represented as follows

$$I_c(\Phi, \rho) = I(B:R)_{\Phi \otimes \text{Id}_R}(\hat{\rho}) - H(\rho),$$

where $\hat{\rho}$ is a purification in $\mathfrak{S}(\mathcal{H}_{AR})$ of a state ρ . Hence for arbitrary quantum channels Φ and Ψ , any n and a state ρ in $\mathfrak{S}(\mathcal{H}_A^{\otimes n})$ we have

$$I_c(\Phi^{\otimes n}, \rho) - I_c(\Psi^{\otimes n}, \rho) = I(B^n:R^n)_{\Phi^{\otimes n} \otimes \text{Id}_{R^n}}(\hat{\rho}) - I(B^n:R^n)_{\Psi^{\otimes n} \otimes \text{Id}_{R^n}}(\hat{\rho}) \quad (65)$$

where $\hat{\rho}$ is a purification in $\mathfrak{S}(\mathcal{H}_{AR}^{\otimes n})$ of the state ρ . This representation, Proposition 3 in Section 4.2 and Lemma 12 in [13] imply (62).

Continuity bound (63) is obtained by using Proposition 5 in Section 5.1 twice and by noting that $\beta(\hat{\Phi}, \hat{\Psi}) = \beta(\Phi, \Psi)$.

To prove continuity bound (64) note that representation (22) implies

$$\bar{C}_p(\Phi^{\otimes n}) = \sup_{\hat{\rho}} \left[I(B^n:C)_{\Phi^{\otimes n} \otimes \text{Id}_C}(\hat{\rho}) - I(E^n:C)_{\hat{\Phi}^{\otimes n} \otimes \text{Id}_C}(\hat{\rho}) \right], \quad (66)$$

¹²We assume that expressions (53)-(59) remain valid in the case $\dim \mathcal{H}_B = +\infty$.

where the supremum is over all qc -states in $A^n C$. Since $\beta(\widehat{\Phi}, \widehat{\Psi}) = \beta(\Phi, \Psi)$, inequality (64) is obtained by using Proposition 3 in Section 4.2 twice and Lemma 12 in [13].

To show the tightness of continuity bounds (60), (62) and (63) consider the family of erasure channels

$$\Phi_p(\rho) = \begin{bmatrix} (1-p)\rho & 0 \\ 0 & p\text{Tr}\rho \end{bmatrix}, \quad p \in [0, 1].$$

from d -dimensional system A to $(d+1)$ -dimensional system B . It is well known (see [7, 26]) that

$$C(\Phi_p) = \bar{C}(\Phi_p) = (1-p) \log d \quad (67)$$

and

$$Q(\Phi_p) = C_p(\Phi_p) = \bar{C}_p(\Phi_p) = \max\{(1-2p) \log d, 0\}. \quad (68)$$

By writing the channel Φ_p as the map $\rho \mapsto (1-p)\rho \oplus [p\text{Tr}\rho]|\psi\rangle\langle\psi|$ from $\mathfrak{T}(\mathcal{H}_A)$ to $\mathfrak{T}(\mathcal{H}_A \oplus \mathcal{H}_\psi)$, where \mathcal{H}_ψ is the space generated by $|\psi\rangle$, we see that the isometry

$$V_p : |\varphi\rangle \mapsto \sqrt{1-p}|\varphi\rangle \otimes |\psi\rangle \oplus \sqrt{p}|\psi\rangle \otimes |\varphi\rangle$$

from \mathcal{H}_A into \mathcal{H}_{BE} , where $\mathcal{H}_E = \mathcal{H}_B = \mathcal{H}_A \oplus \mathcal{H}_\psi$, is a Stinespring isometry for Φ_p , i.e. $\Phi_p(\rho) = \text{Tr}_E V_p \rho V_p^*$, for each p . Direct calculation shows that

$$\|V_{1/2-x} - V_{1/2}\| = \sqrt{2 - \sqrt{1-2x} - \sqrt{1+2x}} = x + o(x) \quad (x \rightarrow 0). \quad (69)$$

It follows from (67) and (68) that $\bar{C}(\Phi_{1/2-x}) - \bar{C}(\Phi_{1/2}) = x \log d$ and that

$$Q(\Phi_{1/2-x}) - Q(\Phi_{1/2}) = \bar{C}_p(\Phi_{1/2-x}) - \bar{C}_p(\Phi_{1/2}) = 2x \log d.$$

Since (69) implies $\beta(\Phi_{1/2-x}, \Phi_{1/2}) \leq x + o(x)$ for small x , we see that continuity bounds (60), (62) and (63) are tight (for large d_A).

The last assertion of the proposition follows from the right inequality in (6) and monotonicity of the function $g(x)$. \square

7 Continuity bounds for energy-constrained capacities of infinite-dimensional channels.

When we consider transmission of information over infinite-dimensional quantum channels we have to impose constraints on states used for coding information to be consistent with the physical implementation of the process. A typical physically motivated constraint is the requirement of bounded average energy of states used for coding information. For a single channel this constraint is expressed by the inequality

$$\mathrm{Tr} H_A \rho \leq E, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \quad (70)$$

where H_A is the Hamiltonian of input quantum system A , in the case of n -copies of a channel it can be written as follows

$$\mathrm{Tr} H_A^n \rho \leq nE, \quad \rho \in \mathfrak{S}(\mathcal{H}_A^{\otimes n}), \quad (71)$$

where $H_A^n = H_A \otimes I_A \otimes \dots \otimes I_A + \dots + I_A \otimes \dots \otimes I_A \otimes H_A$ is the Hamiltonian of the system A^n [7, 8, 27].

We will assume that the Hamiltonian H_A satisfies condition (16).

The Holevo capacity of any channel $\Phi : A \rightarrow B$ with constraint (71) can be defined as follows:

$$\bar{C}(\Phi, H_A, E) = \sup_{\mathrm{Tr} H_A \bar{\rho}(\mu) \leq E} \chi(\Phi(\mu)), \quad (72)$$

where $\chi(\Phi(\mu))$ is the output Holevo quantity of an ensemble μ and the supremum is over all ensembles in $\mathcal{P}(\mathcal{H}_A)$ with the average energy $\leq E$.¹³

By the Holevo-Schumacher-Westmoreland theorem adapted for constrained infinite-dimensional channels (see [8]) the classical capacity of any channel $\Phi : A \rightarrow B$ with constraint (71) is given by the regularized expression

$$C(\Phi, H_A, E) = \lim_{n \rightarrow +\infty} n^{-1} \bar{C}(\Phi^{\otimes n}, H_A^n, nE).$$

By the Bennett-Shor-Smolín-Thapalyal theorem adapted for constrained infinite-dimensional channels (see [8, 10]) the classical entanglement-assisted

¹³The suprema in (72) and in expression (73) below can be taken only over discrete ensembles satisfying the energy constraint [9].

capacity of any channel $\Phi : A \rightarrow B$ with constraint (71) is given by the expression

$$C_{\text{ea}}(\Phi, H_A, E) = \sup_{\text{Tr} H_A \rho \leq E} I(\Phi, \rho),$$

where $I(\Phi, \rho)$ is the quantum mutual information defined in (56).

Detailed analysis of the energy-constrained quantum and private capacities in the context of general-type infinite-dimensional channels¹⁴ has been made recently by Wilde and Qi in [27]. The results in [27] and [28] give considerable reasons to conjecture validity of the following generalizations of the Lloyd-Devetak-Shor theorem and of the Devetak theorem to constrained infinite-dimensional channels:

- the quantum capacity of any channel $\Phi : A \rightarrow B$ with constraint (71) is given by the regularized expression

$$Q(\Phi, H_A, E) = \lim_{n \rightarrow +\infty} n^{-1} \bar{Q}(\Phi^{\otimes n}, H_A^n, nE),$$

where $\bar{Q}(\Phi, H_A, E)$ is the least upper bound of the coherent information $I_c(\Phi, \rho) \doteq I(\Phi, \rho) - H(\rho)$ on the set of all input states $\rho \in \mathfrak{S}(\mathcal{H}_A)$ satisfying (70).

- the private capacity of any channel $\Phi : A \rightarrow B$ with constraint (71) is given by the regularized expression

$$C_p(\Phi, H_A, E) = \lim_{n \rightarrow +\infty} n^{-1} \bar{C}_p(\Phi^{\otimes n}, H_A^n, nE),$$

where

$$\bar{C}_p(\Phi, H_A, E) = \sup_{\text{Tr} H_A \bar{\rho}(\mu) \leq E} \left[\chi(\Phi(\mu)) - \chi(\hat{\Phi}(\mu)) \right] \quad (73)$$

(the supremum is over all ensembles in $\mathcal{P}(\mathcal{H}_A)$ with the average energy not exceeding E and $\hat{\Phi}$ is the complementary channel to the channel Φ defined in (2)).

Winter's type continuity bound for the capacity $C_{\text{ea}}(\Phi, H_A, E)$ is obtained in [22, Pr.11]. By using expression (65) with $n = 1$ it is easy to show

¹⁴There are many papers devoted to analysis of these capacities for Gaussian channels, see [11, 31] and the surveys in [24, 27].

that the same continuity bound holds for $\bar{Q}(\Phi, H_A, E)$. It follows that the functions

$$\Phi \mapsto C_{\text{ea}}(\Phi, H_A, E) \quad \text{and} \quad \Phi \mapsto \bar{Q}(\Phi, H_A, E)$$

are uniformly continuous on the set of all channels from a given infinite-dimensional system A to arbitrary system B with respect to the diamond norm topology provided that the Hamiltonian H_A satisfies condition (16).

Similar assertions for other basic capacities under slightly stronger condition on the Hamiltonian H_A are presented in the following theorem.

Theorem 2. *If the Hamiltonian H_A of input system A satisfies condition (19) then for any $E > E_0$ all the functions*

$$\Phi \mapsto X(\Phi, H_A, E), \quad X = \bar{C}, C, Q, \bar{C}_p, C_p,$$

are uniformly continuous on the set of all channels from A to arbitrary system B with respect to the Bures distance (the diamond norm). Moreover,

$$|\bar{C}(\Phi, H_A, E) - \bar{C}(\Psi, H_A, E)| \leq C(\varepsilon, E) + 2g(\varepsilon), \quad (74)$$

$$|C(\Phi, H_A, E) - C(\Psi, H_A, E)| \leq C(\varepsilon, E) + 2g(\varepsilon), \quad (75)$$

$$|Q(\Phi, H_A, E) - Q(\Psi, H_A, E)| \leq C_*(\varepsilon, E) + 2g(\varepsilon), \quad (76)$$

$$|\bar{C}_p(\Phi, H_A, E) - \bar{C}_p(\Psi, H_A, E)| \leq 2C(\varepsilon, E) + 4g(\varepsilon), \quad (77)$$

$$|C_p(\Phi, H_A, E) - C_p(\Psi, H_A, E)| \leq 2C(\varepsilon, E) + 4g(\varepsilon), \quad (78)$$

for any channels $\Phi : A \rightarrow B$ and $\Psi : A \rightarrow B$, where $\varepsilon = \beta(\Phi, \Psi) \leq \|\Phi - \Psi\|_{\diamond}^{1/2}$, $C(\varepsilon, E)$ and $C_(\varepsilon, E)$ are quantities defined, respectively, in (42) and (43), which tend to zero as $\varepsilon \rightarrow +0$, and $g(\varepsilon) = (1 + \varepsilon)h_2(\frac{\varepsilon}{1+\varepsilon})$.*

If A is the ℓ -mode quantum oscillator and $\varepsilon \leq 1$ then the quantities $C(\varepsilon, E)$ and $C_(\varepsilon, E)$ in the above inequalities can be replaced by their upper bounds $\hat{C}(\varepsilon, E)$ and $\hat{C}_*(\varepsilon, E)$ defined, respectively, in (44) and (45).*

Proof. Continuity bound (74) for the Holevo capacity directly follows from its definition (72) and Proposition 7 in Section 5.2.

Continuity bound (75) is proved by using representation (22), Proposition 4 in Section 4.2 and Lemma 12 in [13].

Continuity bound (76) is proved by using expression (65), Proposition 4 in Section 4.2 and Lemma 12 in [13].

Continuity bound (77) is obtained by using Proposition 7 twice and by noting that $\beta(\widehat{\Phi}, \widehat{\Psi}) = \beta(\Phi, \Psi)$.

Continuity bound (78) is proved by using the constrained version of representation (66), Proposition 4 twice and Lemma 12 in [13] and by noting that $\beta(\widehat{\Phi}, \widehat{\Psi}) = \beta(\Phi, \Psi)$.

The last assertion of the theorem follows from Remark 2. \square

8 Open problem

Using the extended conditional mutual information and the modification of the Alicki-Fannes-Winter method adapted for sets of states with bounded energy we have shown uniform continuity of the basic capacities under the input energy constraint *on the set of all quantum channels equipped with the diamond norm* (equivalent to the Bures distance) provided that the Hamiltonian H_A of input system satisfies condition (19). But in infinite dimensions along with the topology induced by the diamond norm one can consider weaker topologies on the set of quantum channels, in particular, the *strong convergence topology* generated by the strong operator topology on the set of all linear bounded operators between the Banach spaces $\mathfrak{I}(\mathcal{H}_A)$ and $\mathfrak{I}(\mathcal{H}_B)$. The strong convergence of a sequence $\{\Phi_n\}$ of quantum channels to a quantum channel Φ_0 means that

$$\lim_{n \rightarrow +\infty} \Phi_n(\rho) = \Phi_0(\rho) \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_A).$$

Use of the strong convergence topology in infinite dimensions seems preferable, since it describes, roughly speaking, perturbations of a channel corresponding to deformation of its Stinespring isometry in the strong operator topology, while closeness of two channels in the diamond norm means, by inequality (6), the operator norm closeness of their Stinespring isometries. In other words, the diamond norm topology is too strong to describe all physical perturbations of a channel. This is confirmed by examples of channels with close parameters having large diamond norm of the difference [30].

So, it seems very desirable to strengthen the main assertion of Theorem 2 by proving continuity of the basic capacities with respect to the strong convergence topology. A partial result in this direction is Proposition 11 in [21] which asserts continuity of the function $\Phi \mapsto C_{\text{ea}}(\Phi, H_A, E)$ on the set of all channels with respect to the strong convergence topology if the Hamiltonian H_A of input system satisfies condition (16).

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